Optimal Multi-Degree Reduction of Bézier Curves with Constraints, Using Matrix Computations

Davood Bakhshesh and Mohammad Reza Samiee

***Abstract*—In this paper, we consider multi-degree reduction of Bézier curves with constraints of endpoints continuity with respect to L_2 norm, using matrix computations. We find an explicit form of the multi-degree reduction matrix for Bézier curve with constraints of endpoints continuity.**

***Index Terms*—Bézier curves, Degree reduction, Degree elevation, Endpoints continuity.**

I. Introduction

The exchanging of product model data between various CAD/CAM systems is often needed. However the representation schemes of parametric curves and surfaces are varied in different geometric modeling systems. Such as, the maximum degree, which different computer systems can deal with, varies quite dramatically. Therefore for the data communication between diverse CAD/CAM systems, curves of high degree must be approximated by curves of lower degree due to variation in the maximum degree allowed. Thus the problem of how to optimally approximate a given parametric curve by a lower degree curve within a certain error bound has arisen in CAGD.

In recent years, many methods have been used to reduce the degree of Bézier curves. The problem of degree reduction is viewed as the inverse process of degree elevation (Forrest, 1972; Farin, 1983; Piegl and Tiller, 1995). In general, degree reduction is not exactly possible in contract to the reverse process of degree elevation. Thus degree reduction approximation of parametric curves and surfaces has been widely studied. Discrete points and derivative information of original curve are used in degree reduction approximation [1]. The degree reduction of Bézier curves can also been done by using Chebyshev polynomials approximation (Watkins and Worsey, 1988; Lachance, 1988). A simple geometric constructive method of degree reduction with constrained Chebyshev polynomials is presented in [2]; while a least squares method of degree reduction with constrained Legendre polynomials is presented in [3]. Using conversion of bases between Chebyshev and Bernstein bases, a method of degree reduction with the reduction matrix is developed [4]. From the practical point of view, when transmitting geometric information from one system to another, it is our general aim to ensure a high degree of accuracy and the least possible loss of geometric information. Moreover degree reduction schemes often need to be combined with the subdivision algorithm, i.e., a high degree curve is approximated by a number of lower degree curve segments and continuity between adjacent lower degree curve segments should be maintained. Unfortunately, all methods known up-to-now have some disadvantages. First, they have no explicit solutions for optimal multi-degree reduction with constraints of endpoints continuity of high order and have to be determined by numeric algorithms such as Reme-stype algorithm. Secondly, for the multi-degree reduction, most methods need stepwise approximation and hence a lot of time for computing is spent. Thirdly, most methods in general cannot achieve the optimal approximation any more. In this paper, we study the multi-degree reduction of Bézier curves with constraints, using matrix computations. The organization of the paper is as follows: We introduce some basic results in Section 2. Optimal multi-degree reduction of Bézier curves, using matrix computations in Section 3. Finally section 4 concludes this paper.

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II. Preliminaries

A. Bézier Curves

The Bézier representation of a parametric polynomial curve $P_n(t)$ of degree n is expressed using the Bernstein polynomials as basis in the form:

$$P_n(t) = \sum_{i=0}^{n} p_i B_i^n(t), \qquad 0 \le t \le 1, \tag{1}$$

where the $\{p_i\}_{i=0}^{n}$ is the set of $(n+1)$ Bézier points, and

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \qquad i = 0,1,\cdots,n$$

Are the Bernstein polynomials of degree n.

The area under a Bernstein polynomial $B_k^n(t), k = 0,1,\ldots,n$ of degree n is given by

$$\int_0^1 B_k^n(t)dt = \frac{1}{n+1} \tag{2}$$

The product of Bernstein polynomials of degree n and m is also a Bernstein polynomial of degree $n + m$ and given by

$$B_i^n(t) B_j^m(t) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}} B_{i+j}^{n+m}(t). \tag{3}$$

The optimal multi-degree reduction with constraints of endpoints continuity is defined as follows

Definition 1 Given a degree n Bézier curve $P_n(t)$, the optimal multi-degree reduction with constraints of endpoints continuity of orders $r, s, (r, s \geq 0)$ is to find a degree $m(m < n-1)$ Bézier curve $Q_m(t) = \sum_{i=0}^{m} q_i B_i^m(t)$, such that

$$\|P_n - Q_m\|_{L_2} = \int_0^1 t^\alpha (1-t)^\beta |P_n(t) - Q_m(t)|^2 dt \quad (\alpha, \beta > -1) \tag{4}$$

is minimized and

$$\frac{d^i Q_m(0)}{dt^i} = \frac{d^i P_n(0)}{dt^i}, \qquad i = 0,1,\cdots,r;$$
$$\frac{d^j Q_m(1)}{dt^j} = \frac{d^j P_n(1)}{dt^j}, \qquad j = 0,1,\cdots,s.$$

The following theorem presents a key idea to obtain the optimal multi-degree reduced Bézier curve with constraints of endpoints continuity.

Theorem 1 (See [5]) Given a degree n Bézier curve $P_n(t) = \sum_{i=0}^{n} p_i B_i^n(t), t\epsilon[0,1]$, and let $r + s < m < n - 1$, and $r, s \leq n - m$, then the curve can be expressed as

$$P_n(t) = \sum_{i=0}^{r} q_i B_i^m(t) + \sum_{i=r+1}^{n-s-1} p_i^I B_i^n(t) + \sum_{i=m-s}^{m} q_i B_i^m(t), \tag{5}$$

where q_i's and p^I's are defined as follows:

$$\begin{cases} q_0 = \dfrac{1}{b_{0,0}^{(m,n)}} p_0, \\ q_j = \dfrac{1}{b_{j,j}^{(m,n)}} \left(p_j - \sum_{i=0}^{j-1} b_{i,j}^{(m,n)} q_i \right), \quad j = 1,2,\cdots,r, \\ q_m = \dfrac{1}{b_{m,n}^{(m,n)}} p_n, \\ q_{m-j} = \dfrac{1}{b_{m-j,n-j}^{(m,n)}} \left(p_{n-j} - \sum_{i=0}^{j-1} b_{m-i,n-j}^{(m,n)} q_{m-i} \right), j = 1,2,\cdots,s, \end{cases} \tag{6}$$

And

$$b_{m,n}^{(m,n)} = \binom{m}{i}\binom{n-m}{j-i} \Big/ \binom{n}{j}. \tag{7}$$

When $m - s > n + r - m$,

$$\begin{cases} p_j^I = p_j - \sum_{i=\max(0,j-(n-m))}^{r} b_{m,n}^{(m,n)} q_i, \\ \qquad j = r+1, r+2, \cdots, n+r-m, \\ p_j^I = p_j \\ j = n+r-m+1, n+r-m+2, \cdots, m-s-1, \\ p_j^I = p_j - \sum_{i=\max(0,j-m)}^{s} b_{m-i,j}^{(m,n)} q_{m-i}, \\ \qquad j = m-s, \cdots, n-s-1. \end{cases} \tag{8}$$

When $m - s \leq n + r - m$,

$$\begin{cases} p_j^I = p_j - \sum_{i=\max(0,j-(n-m))}^{r} b_{i,j}^{(m,n)} q_i, \\ \qquad j = r+1, r+2, \cdots, m-s-1, \\ p_j^I = p_j - \sum_{i=\max(0,j-(n-m))}^{r} b_{i,j}^{(m,n)} q_i - \sum_{i=\max(0,m-j)}^{s} b_{m-i,j}^{(m,n)} q_{m-i}, \\ \qquad j = m-s, m-s+1, \cdots, n+r-m, \\ p_j^I = p_j - \sum_{i=\max(0,m-j)}^{s} b_{m-i,j}^{(m,n)} q_{m-i}, \\ \qquad j = n+r-m+1, \cdots, n-s-1. \end{cases} \tag{9}$$

Also, $\{q_i\}_{i=0}^{r}$ and $\{q_i\}_{i=m-s}^{m}$ are the part of the control points of the degree reduced curve $Q_m(t)$ of degree m satisfying

$$\begin{aligned} \frac{d^\lambda Q_m(0)}{dt^\lambda} &= \frac{d^\lambda P_n(0)}{dt^\lambda}, \qquad \lambda = 0,1,\cdots,r; \\ \frac{d^\pi Q_m(0)}{dt^\mu} &= \frac{d^\mu P_n(0)}{dt^\mu}, \qquad \mu = 0,1,\cdots,s; \end{aligned} \tag{10}$$

B. L_2-Norm and Degree Elevation

The L_2 norm of the Bézier curve P_n in the Bernstein basis form is given by

$$\|P_n\|_{L_2} = \int_0^1 t^\alpha (1-t)^\beta \left| \sum_{i=0}^{n} p_i B_i^n(t) \right|^2 dt \tag{11}$$

Simplifying this using (2) and (3) gives

$$\|P_n\|_{L_2} = p^T H_n p, \tag{12}$$

where

$$H_n(i,j) = \binom{n}{i}\binom{n}{j} B(\alpha + 2n - i - j, \beta + i + j + 1) \tag{13}$$
$$i, j = 0,1,\cdots,n,$$

and

$$p = (p_0, p_1, \cdots, p_n)^T,$$
$$B(x,y) = \Gamma(x+1)\Gamma(y+1)/\Gamma(x+y+2).$$

The matrix H_n is a real symmetric matrix, as a consequence of the symmetry of the combinatorial function. The matrix H_n is also a positive definite matrix, as a consequence of the positivity of the left-hand side in the definition. Thus, H_n is a real symmetric positive definite matrix.

Now, we want to express the degree elevation of Bézier curve problem in matrix representation.

The degree elevation problem is concerned with writing a given Bézier curve in basis of degree n into basis of degree $n+1$ without changing the curve. The new vertices $p_i^{(1)}$ of the new polygon are calculated from the following formula, see [6]:

$$p_i^{(1)} = \frac{i}{n+1} p_{i-1} + \left(1 - \frac{i}{n+1}\right) p_i, i = 0,1,\cdots,n+1. \tag{14}$$

The matrix form of (14) is given by $p^{(1)} = T_n p$, where

$p = (p_0, p_1, \cdots, p_n)^T, p^{(1)} = \left(p_0^{(1)}, p_1^{(1)}, \cdots, p_{n+1}^{(1)}\right)^T$, and the $(n+2) \times (n+1)$ matrix T_n is given by

$$T_n = \frac{1}{n+1}\begin{pmatrix} n+1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & n & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & n-1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & n+1 \end{pmatrix}$$

This process can be repeated c times to obtain a new sequence of control points $p^{(c)}$ for the degree elevated curve, where $p^{(c)} = T_{n,c}f$, and the $(n + c + 1) \times (n + 1)$ matrix $T_{n,c}$ is given by

$$T_{n,c} = T_{n+c-1}T_{n+c-2} \cdots T_{n+1}T_n$$

And has the elements

$$T_{n,c} = \frac{\binom{n}{j}\binom{c}{i-j}}{\binom{n+c}{i}}, \quad i = 0,1,\cdots,n+c \ \ and \ \ j = 0,1,\cdots,n.$$

III. Optimal Multi-Degree Reduction of Bézier Curve

Let $\bar{q} = (q_0, q_1, \cdots, q_r, 0,0, \cdots ,0, q_{m-s}, \cdots, q_m)^T$. As shown in Eq. (6), the q_i's in $\bar{q}$ are expressed recursively. We can find a matrix $\bar{Q}$ such that

$$\bar{q} = \bar{Q}p. \tag{15}$$

Theorem 2 (See [7]) $\bar{q} = \bar{Q}p$ where $\bar{Q} = (q_{jk})_{(m+1)\times(n+1)}$ and q_{jk} is defined by

$$q_{jk} = \begin{cases} \dfrac{\binom{n}{k}}{\binom{m}{j}} a_{j-k}, & j = 0,1,\cdots,r; k = 0,1,\cdots,j, \\ \dfrac{\binom{n}{k}}{\binom{m}{j}} a_{k-j-(n-m)}, & \begin{cases} j = m-s, m-s+1, \cdots, m; \\ k = j+(n-m), \cdots, n, \end{cases} \\ 0, & elsewhere. \end{cases} \tag{16}$$

And

$$\begin{cases} a_0 = 1, \\ a_k = -\displaystyle\sum_{i=0}^{k-1}\binom{n-m}{k-i} a_i, \ \ k = 1,2,\cdots,r, \ \ or \ \ k = 1,2,\cdots,s \end{cases} \tag{17}$$

Now, we want to find $\{q_i\}_{i=r+1}^{m-s-1}$. Denote $P_n^I(t) = \sum_{i=r+1}^{n-s-1} p_i^I B_i^n(t)$ and $p_i^I = q_i \ (i = r+1, \cdots, m-s-1)$. Then the problem of optimal multi-degree reduction of $P_n(t)$ is equal to finding $Q_m^I(t) = \sum_{i=r+1}^{m-s-1} q_i^I B_i^m(t)$, such that $\|P_n^I - Q_m^I\|_{L_2}$, is minimized. (Without constraints of endpoints continuity)

By the properties of Bernstein polynomials, we ha$P_n^I(t) = \sum_{i=r+1}^{n-s-1} p_i^I B_i^n(t) = (1-t)^{s+1}t^{r+1}\sum_{i=0}^{N} p_i^{II} B_i^N(t)$

$$= (1-t)^{s+1}t^{r+1}P_N^{II}(t), \tag{18}$$

where

$$N = n - (r+s+2),$$
$$p_i^{II} = p_{r+1+i}^{II}\binom{n}{r+1+i}/\binom{N}{i}. \ i = 0,1,\cdots,N.$$

Using the similar relations, we have

$$Q_m^I(t) = (1-t)^{s+1}t^{r+1}\sum_{i=0}^{M} q_i^{II} B_i^M(t)$$
$$= (1-t)^{s+1}t^{r+1}Q_M^{II}(t), \tag{19}$$

where

$$M = m - (r+s+2),$$
$$q_i^{II} = q_{r+1+i}^{II}\binom{m}{r+1+i}/\binom{M}{i}, \ i = 0,1,\cdots,M. \tag{20}$$

Using the (18) and (19) it is easy to know the problem of optimal multi-degree reduction of $P_n(t)$ is equal to finding $Q_M^{II}(t)$ such that $\|P_N^{II} - Q_M^{II}\|_{L_2}$, is minimized. (Without constraints of endpoints continuity)

By the subsection B of section 2 , elevating the degree of Q_M^{II} from M to N using the matrix $T_{M,c}$, where $c = N - M$, gives

$$q^{(c)} = T_{M,c}q^{II},$$

where $q^{II} = (q_0^{II}, q_1^{II}, \cdots, q_M^{II})^T$.

This last step rewrites the curve Q_M^{II} of degree M as a curve of degree N

$$Q_M^{II}(t) = \sum_{i=0}^{N} q_i^{(c)} B_i^N(t),$$

And hence, the L_2-norm is given by

$$\|P_N^{II} - Q_M^{II}\|_{L_2} = \int_0^1 t^\alpha(1-t)^\beta \left|\sum_{i=0}^{N}\left(q_i^{(c)} - p_i^{II}\right)B_i^N(t)\right|^2 dt$$

Invoking (12) into the last equation gives the L_2-norm between the Bézier curves P_N^{II} and Q_M^{II} in the following formula:

$$\|P_N^{II} - Q_M^{II}\|_{L_2} = V^T H_N V, \tag{21}$$

where

$$V = p^{II} - T_{M,c}q^{II}, p^{II} = (p_0^{II}, p_1^{II}, \cdots, p_N^{II})^T,$$
$$q^{II} = (q_0^{II}, q_1^{II}, \cdots, q_M^{II})^T.$$

Substituting $V = p^{II} - T_{M,c}q^{II}$ in $\|P_N^{II} - Q_M^{II}\|_{L_2}$ and doing some algebraic manipulations gives:

$$\|P_N^{II} - Q_M^{II}\|_{L_2} = p^{II^T} H_N p^{II} - 2q^{II^T} T_{M,c}^T H_N p^{II} + q^{II^T} T_{M,c}^T H_N T_{M,c} q^{II}. \tag{22}$$

The error, defined above, is a function of the elements of the vectorq^{II}. To find the minimum, we use the method of least squares approximation to find the vector $\hat{q}^{II}$ that minimizes the last formula. We insist that the first partial derivatives $\frac{\partial (V^T H_N V)}{\partial q^{II}}$ are equal zero. This process leads to the normal equations:

$$T_{M,c}^T H_N T_{M,c} \hat{q}^{II} = T_{M,c}^T H_N p^{II}. \tag{23}$$

Theorem 3 the $(n + 1) \times (n + 1)$ matrix $T_{n-1}^T H_n T_{n-1}$ has the following property:

$$T_{n-1}^T H_n T_{n-1} = H_{n-1} \tag{24}$$

Proof

$$\begin{aligned} & T_{n-1}^T H_n T_{n-1}(i,j) \\ &= \sum_{l=0}^{n+1} \frac{\binom{n-1}{i}\binom{1}{l-j}}{\binom{n}{l}} \sum_{k=0}^{n+1} \left\{ \binom{n}{k}\binom{n}{l} \times B(\alpha + 2n - l - k + 1, \beta + l + k + 1) \frac{\binom{n-1}{j}\binom{1}{k-j}}{\binom{n}{k}} \right\} \\ &= \binom{n-1}{i}\binom{n-1}{j} \sum_{l=0}^{n+1}\sum_{k=0}^{n+1} \binom{1}{l-j}\binom{1}{k-j} B(\alpha + 2n - l - k + 1, \beta + l + k + 1) \end{aligned}$$

Using $\Gamma(x + 1) = x\Gamma(x)$ in the last equation, we have

$$\begin{aligned} & T_{n-1}^T H_n T_{n-1}(i,j) \\ &= \binom{n-1}{i}\binom{n-1}{j} B(\alpha + 2n - l - k + 1, \beta + l + k + 1) \\ &= H_{n-1}. \square \end{aligned}$$

From Theorem 3, we have $T_{M,c}^T H_N T_{M,c} = H_M$. Hence, the real symmetric positive definite $T_{M,c}^T H_N T_{M,c}$ is invertible. Hence, the normal equations are uniquely solvable and have the solution:

$$\hat{q}^{II} = H_M^{-1} T_{M,c}^T H_N p^{II}. \tag{25}$$

Using (20), we have

$$q_j = q_j^I = \hat{q}_{j-r-1}^{II} \binom{M}{j-r-1} \Big/ \binom{m}{j} \qquad j = r+1, r+2, \cdots, m-s-1. \tag{26}$$

IV. Conclusion

We have proposed an approach to the problem of optimal multi-degree reduction of Bézier curves with constraints, using matrix computations with respect to L_2-norm. The approach has described in this paper is based on the results of Chen and Wang (2002) (See [5]). The approach in this paper can be effectively combined with the subdivision algorithm for the better approximation of degree reduction.

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