The Weighted Dual Functions for Wang-Said Type Generalized Ball Bases With and Without Boundary Constraints

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Abstract—By introducing the inner-product matrix of two vector functions and using conversion matrix, explicit formulas for the dual basis functions of Wang-Said type generalized Ball bases (WSGB) with respect to the Jacobi weight function are given. The dual basis functions with and without boundary constraints are also considered. As a result, the paper includes the weighted dual basis functions of Bernstein basis, Wang-Ball basis and some intermediate bases. These results are useful for the application of generalized Ball curves of Wang-Said type and their popularization in Computer Aided Geometric Design.

Index Terms—Weighted dual function, WSGB, inner-product matrix, bernstein basis

I. INTRODUCTION

Over the past several decades, there have been a few studies on generalized Ball curves. The cubic Ball curves were first introduced by Ball (1974) in a lofting surface program CONSURF at the British Aircraft Corporation [1], [2]. Since then, several researchers [3]-[9] have investigated the generalized Ball curves and theoretically come to the conclusion that generalized Ball curves are more efficient than Bézier curves in calculation, the degree elevation and reduction, etc. In [10], [11], Wu presented two new families of generalized Ball curves. One is generalized Ball curves of Said- Bézier type, and the other is generalized Ball curves of Wang-Said type (WSGB curves). Then the authors ([9]) worked out the dual functionals and the relevant basis transformation formulae for the generalized Ball basis of Wang-Said type.

The Bernstein polynomials have many important properties which make them the most commonly used basis in approximation theory and CAGD [12], [13], but they are not orthogonal and not appropriate to be used in the least square approximation. To overcome this difficulty, Jüttler [14] first gave an explicit formula for the dual basis functions of the Bernstein polynomials with respect to the usual inner product of Hilbert space. Rababah and Al- Natour [15], [16] generalized these results to the dual basis functions of the Bernstein polynomials with respect to the Jacobi weight function, and he also considered dual functions with and without boundary constraints. By introducing the inner-product matrix of two vector functions and using conversion matrix, we give explicit formulas for the dual

basis functions of Wang-Said type generalized Ball bases (WSGB) with respect to the Jacobi weight function. The structure of this paper is as follows: in Section 2 we introduce WSGB and conversion matrices from WSGB to Bernstein basis. The weighted dual functions of WSGB with and without boundary constraints are introduced in Section 3. Finally, we end this paper with conclusions in Section 4.

II. WSGB AND CONVERSION MATRIX FROM WSGB TO BERNSTEIN BASIS

Suppose P^n is the $(n + 1)$ -dimensional real linear space of all polynomials of degree at most n . We focus on two types of bases: WSGB and Bernstein basis. The definition of WSGB is as follows:

Definition 1. (see [17]). Suppose m is a natural number, $n = 2m + 1$ or $n = 2m$, and $L(0 \leq L \leq m - 1)$ is the given position parameter,

$$\beta_i(t; n, L) = \begin{cases} \binom{\lfloor n/2 \rfloor - L + i}{i} t^i (1 - t)^{\lfloor n/2 \rfloor - L + 1}, & 0 \leq i \leq \lfloor n/2 \rfloor - L - 1, \\ 2^{i - \lfloor n/2 \rfloor + L} \binom{\lfloor n/2 \rfloor - 2L}{\lfloor n/2 \rfloor - L} t^i (1 - t)^{i+2}, & \lfloor n/2 \rfloor - L \leq i \leq \lfloor n/2 \rfloor - 1, \\ 2^L \binom{2\lfloor n/2 \rfloor - 2L}{\lfloor n/2 \rfloor - L} t^{\lfloor n/2 \rfloor} (1 - t)^{\lfloor n/2 \rfloor}, & i = \lfloor n/2 \rfloor, \\ 2^L \binom{2\lfloor n/2 \rfloor - 2L}{\lfloor n/2 \rfloor - L} (1 - t)^{\lfloor n/2 \rfloor} t^{\lfloor n/2 \rfloor}, & i = \lfloor n/2 \rfloor, \\ \beta_{n-i}(1 - t; n, L), & \lfloor n/2 \rfloor + 1 \leq i \leq n, \end{cases} \quad (1)$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the greatest integer less than or equal to x and the least integer greater than or equal to x respectively. Then $\{\beta_i(t; n, L)\}_{i=0}^n$ are defined as Wang-Said type generalized Ball bases (WSGB). Obviously, WSGB basis unify the Said- Ball basis ($L = 0$) and Wang-Ball basis ($L = m - 1$) and include some intermediate basis ($L = 1, 2, \dots, m - 2$).

Let $\{p_i\}_{i=0}^n$ be a set of control points in $\mathbb{R}^2$ or $\mathbb{R}^3$. Using bases $\{\beta_i(t; n, L)\}_{i=0}^n$, generalized ball curves of Wang- Said type of degree n with position parameter L ($L =$

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$0, 1, \dots, m-1$ are defined as follows

$$r(t; n, L) = \sum_{i=0}^n p_i \beta_i(t; n, L), \quad 0 \leq t \leq 1. \quad (2)$$

Theorem 1. (see [17]). The conversion formulas between the Bernstein basis $B_i^n(u)$ of degree n and the WSGB basis $\beta_i(t; n, L)$ for $i = 0, 1, \dots, n$, can be expressed as follows

$$B_i^n(t) = \sum_{j=0}^n c_{i,j}^{(n)} \beta_j(t; n, L), \quad \beta_i(t; n, L) = \sum_{j=0}^n b_{i,j}^{(n)} B_j^n(t),$$

where ([9], [10])

$$b_{i,j}^{(n)} = b_{2m+1-i, 2m+1-j}^{(n)}$$

$$= \begin{cases} \binom{m-L+i}{i} \binom{m+L-i}{j-i} \binom{2m+1}{j}^{-1}, & 0 \leq i \leq m-L-1, i \leq j \leq m+L; \\ 2^{i-m+L} \binom{2m-2L}{m-L} \binom{2m-1-2i}{j-i} \binom{2m+1}{j}^{-1}, & m-L \leq i \leq m-1; i \leq j \leq 2m-1-2i; \\ 2^L \binom{2m-2L}{m-L} \binom{2m+1}{m}^{-1}, & i = j = m; \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

$$c_{i,j}^{(n)} = \begin{cases} \binom{n}{i} \binom{j - [n/2] - L}{j-i} \binom{m-L+j}{j}^{-1}, & i \leq j \leq m-L-1; \\ 2^{m-j-L} \binom{2m-2L}{m-L}^{-1} \binom{n}{i} \times \\ \left[\binom{2j-n}{j-i} - \sum_{s=m-L-1}^{j-2} 2^{j-2-s} \binom{2s-n+2}{s-i} \right], & m-L \leq j \leq m, j \geq i; \\ 0, & \text{otherwise,} \end{cases} \quad (4)$$

$$c_{i,n-j}^{(n)} = \begin{cases} -2^{m-j-L} \binom{2m-2L}{m-L}^{-1} \binom{n}{i} \times \\ \sum_{s=m-L+1}^{j-2} 2^{j-2-s} \binom{2s-n+2}{s-i}, & m-L+1 \leq j \leq [n/2]-1, j \geq i+2; \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

$$c_{n-i,n-j}^{(n)} = c_{i,j}^{(n)}, \quad (0 \leq i, j \leq n).$$

Suppose

$$\beta(t) = (\beta_0(t; n, L), \beta_1(t; n, L), \dots, \beta_n(t; n, L))^T$$

and

$$B(t) = (B_0^n(t), B_1^n(t), \dots, B_n^n(t))^T$$

Are column vectors of WSGB and Bernstein bases in the linear space P^n respectively. Then the following formulas hold true:

$$B(t) = A\beta(t), \quad \beta(t) = CB(t). \quad (7)$$

Here, $A^{-1} = C = (b_{i,j}^{(n)})_{(n+1) \times (n+1)}$ is defined by (3) and $A = (c_{i,j}^{(n)})_{(n+1) \times (n+1)}$ is defined by (4), (5) and (6).

III. THE WEIGHTED DUAL FUNCTIONS OF WSGB WITH AND WITHOUT BOUNDARY CONSTRAINTS

We first consider the weighted dual functions of WSGB without boundary constraints. The Jacobi weight function $w(t) = (1-t)^\alpha(1+t)^\beta$, $(\alpha, \beta > -1)$ is usually defined in $[-1, 1]$. We use variable substitution: $u = (t+1)/2$ and get the Jacobi weight function on the interval $[0, 1]$ as follows:

$$w(u) = 2^{\alpha+\beta} u^\beta (1-u)^\alpha, \quad u \in [0, 1].$$

The weighted inner product between f and g with respect to the Jacobi weight function is defined as:

$$(f(u), g(u)) = \int_0^1 w(u) f(u) g(u) du.$$

$f(u), g(u) \in P^n$ and $w(u)$ are the Jacobi weight [15]. Then the linear space P^n becomes $(n+1)$ -dimensional Hilbert space. The Gamma function is used to generalize the definition of combinatorial as follows:

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta! (\alpha-\beta)!} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1) \Gamma(\alpha-\beta+1)}, \quad \alpha, \beta > -1$$

Lemma 1. (see [15]). Suppose the dual basis of the Bernstein basis $\{B_i^n(t)\}_{i=0}^n$ with respect to Jacobi weight function is $\{D_i^n(t; \alpha, \beta)\}_{i=0}^n$, which means:

$$(D_i^n(t; \alpha, \beta), B_i^n(t)) = \delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \quad (8)$$

Then $D_p^n(t; \alpha, \beta)$ has the Bernstein representation $D_p^n(t; \alpha, \beta) = \sum_{q=0}^n d_{p,q}^{(\alpha, \beta, n)} B_q^n(t)$ with the real coefficients $d_{p,q}^{(\alpha, \beta, n)} (p, q = 0, 1, \dots, n)$:

$$(6) \quad d_{p,q}^{(\alpha, \beta, n)} = \frac{(-1)^{p+q}}{2^{\alpha+\beta} \binom{n}{p} \binom{n}{q}} \sum_{j=0}^{\min(p,q)} A_j \left[\binom{n+\beta+j+1}{n-p} \binom{n+\alpha-j}{n+\alpha-p} \right. \\ \left. \times \binom{n+\beta+j+1}{n-q} \binom{n+\alpha-j}{n+\alpha-q} \right], \quad (9)$$

where

$$A_j = \frac{\binom{n+\alpha+\beta+j+1}{n+\beta+j+1} (2j+\beta+1)}{\binom{n+\alpha-j}{n-j}}.$$

In order to obtain the matrix expression of the formula (8), we introduce the definition of inner-product matrix as follows.

Definition 2. The inner-product matrix of m dimensional vector function $\lambda(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_m(t))^T$ and n dimensional vector function $\mu(t) = (\mu_1(t), \mu_2(t), \dots, \mu_n(t))^T$ is a $m \times n$ matrix :

$$(\lambda(t), \mu(t)) = \begin{pmatrix} (\lambda_1(t), \mu_1(t)) & (\lambda_1(t), \mu_2(t)) & \dots & (\lambda_1(t), \mu_n(t)) \\ (\lambda_2(t), \mu_1(t)) & (\lambda_2(t), \mu_2(t)) & \dots & (\lambda_2(t), \mu_n(t)) \\ \dots & \dots & \dots & \dots \\ (\lambda_m(t), \mu_1(t)) & (\lambda_m(t), \mu_2(t)) & \dots & (\lambda_m(t), \mu_n(t)) \end{pmatrix} \quad (10)$$

Obviously, inner-product matrix has the following properties:

- 1) Permutability: $(\mu(t), \lambda(t)) = (\lambda(t), \mu(t))^T$,
- 2) Linear property: suppose P, Q are constant matrices.

Then:

$$\begin{aligned} (P\lambda(t), \mu(t)) &= P(\lambda(t), \mu(t)), \\ (\lambda(t), Q\mu(t)) &= (\lambda(t), \mu(t))Q^T. \end{aligned}$$

Suppose vector functions

$$\begin{aligned} D(t) &= (D_0^n(t; \alpha, \beta), D_1^n(t; \alpha, \beta), \dots, D_n^n(t; \alpha, \beta))^T, \\ B(t) &= (B_0^n(t), B_1^n(t), \dots, B_n^n(t))^T, \end{aligned}$$

I is a unit matrix of order $-(n+1)$. Then the formula (8) becomes:

$$(D(t), B(t)) = I.$$

Denote the matrix composed by $d_{p,q}^{(\alpha,\beta,n)}$ as

$F = (d_{p,q}^{(\alpha,\beta,n)})_{(n+1) \times (n+1)}$. Then the formula (8) is equivalent

to $D(t) = FB(t)$. According to the linear property of inner matrix, we have:

$$F(B(t), B(t)) = I. \quad (11)$$

Theorem 2. Suppose the dual basis functions of degree- n WSGB $\{\beta_i(t; n, L)\}_{i=0}^n$ with respect to the Jacobi weight function are $\{\gamma_i(t; n, L)\}_{i=0}^n$, namely:

$$(\gamma_i(t; n, L), \beta_j(t; n, L)) = \delta_{ij}, \quad i, j = 0, 1, \dots, n. \quad (12)$$

Then $\gamma_i(t; n, L)$ can be expressed as

$$\gamma_i(t; n, L) = \sum_{j=0}^n k_{ij} \beta_j(t; n, L), \quad i = 0, 1, \dots, n, \quad (13)$$

where

$$k_{i,j} = \sum_{p=0}^n \sum_{q=0}^n c_{p,i}^{(n)} d_{p,q}^{(\alpha,\beta,n)} c_{q,j}^{(n)}, \quad i, j = 0, 1, \dots, n. \quad (14)$$

Here, $c_{i,j}^{(n)}, d_{i,j}^{(\alpha,\beta,n)}$ are defined by (4), (5), (6) and (9) respectively. The matrix $K = (k_{i,j})_{(n+1) \times (n+1)}$ is defined as the dual matrix of WSGB.

Proof. Replacing $B(t)$ in the formula (11) by (7) and using the linear property of inner matrix, we have

$$\begin{aligned} F(B(t), B(t)) &= \\ F(A\beta(t), A\beta(t)) &= FA(\beta(t), \beta(t))A^T = I, \end{aligned}$$

$$\begin{aligned} FA(\beta(t), \beta(t)) &= (A^T)^{-1}, \\ A^T FA(\beta(t), \beta(t)) &= (A^T FA\beta(t), \beta(t)) = I. \end{aligned}$$

Let $K = A^T FA = (k_{i,j})$. Then vector function

$K(t) = K\beta(t) = (\gamma_0(t; n, L), \gamma_1(t; n, L), \dots, \gamma_n(t; n, L))^T$ satisfies

$$(K(t), \beta(t)) = I,$$

that is

$$(\gamma_i(t; n, L), \beta_j(t; n, L)) = \delta_{ij}, \quad i, j = 0, 1, \dots, n,$$

which means $\{\gamma_i(t; n, L)\}_{i=0}^n$ are dual basis functions of $\{\beta_j(t; n, L)\}_{j=0}^n$ and $\gamma_i(t; n, L) = \sum_{j=0}^n k_{i,j} \beta_j(t; n, L)$, $k_{i,j} = \sum_{p=0}^n \sum_{q=0}^n c_{p,i}^{(n)} d_{p,q}^{(\alpha,\beta,n)} c_{q,j}^{(n)}$.

Let $P^{(n,k,l)}$

$$\begin{aligned} &= \left\{ p(t) \in P^n: \frac{d^i}{dt^i} p(t) \Big|_{t=0} = \frac{d^j}{dt^j} p(t) \Big|_{t=1} = 0 \text{ for } i \right. \\ &\quad \left. = 0, 1, \dots, k-1, j = 0, 1, \dots, l-1 \right\} \quad (15) \end{aligned}$$

be the $(n-k-l+1)$ -dimensional Hilbert space of all polynomials with maximal degree n , whose derivatives of order $\leq k-1$ at $t=0$, as well as derivatives of order $\leq l-1$ at $t=1$, vanish. Obviously, the Bernstein polynomials $\{B_i^n\}_{i=k}^{n-l}$ form a basis for this space. specially, for $k=l=0$ we set $P^{(0,0,0)} = P^n$. Any basis has a

unique dual basis and both the basis and the corresponding dual basis have the same dimensional space, and so the Bernstein basis $\{B_i^n\}_{i=k}^{n-l}$ of $P^{(n,k,l)}$ has a unique dual basis $\{D_i^{(n,k,l)}\}_{i=k}^{n-l}$. The dual basis functions can be represented as linear combinations of Bernstein polynomials as follows.

Theorem 3. The dual $\{D_i^{(n,k,l)}\}_{i=k}^{n-l}$ of the Bernstein basis $\{B_i^n\}_{i=k}^{n-l}$ has the Bernstein representation in the space $P^{(n,k,l)}$:

$$D_p^{(n,k,l)}(t; \alpha, \beta) = \sum_{q=k}^{n-l} d_{i,j}^{(k,l)} B_q^n(t), \quad p = k, \dots, n-l, \quad (16)$$

where the real coefficients $d_{p,q}^{(k,l)}$ satisfy the relation:

$$d_{p,q}^{(k,l)} = \binom{n-k-l}{p-k} \binom{n}{p}^{-1} \binom{n}{q}^{-1} \binom{n-k-l}{q-k} d_{p-k,q-k}^{(\alpha+2l,\beta+2k,n-k-l)}. \quad (17)$$

Proof. It is known (see [19]) that

$$D_p^{(n,k,l)}(t; \alpha, \beta) = \binom{n-k-l}{p-k} \binom{n}{p}^{-1} t^k (1-t)^l D_{p-k}^{n-k-l}(t; \alpha+2l, \beta+2k) \quad (18)$$

using Lemma 1, we have Eq. (19).

$$D_{p-k}^{n-k-l}(t; \alpha+2l, \beta+2k) = \sum_{q=k}^{n-l} d_{p-k,q-k}^{(\alpha+2l,\beta+2k,n-k-l)} B_{q-k}^{n-k-l}(t), \quad (19)$$

with substituting the Eq. (19) and the formula

$$B_{q-k}^{n-k-l}(t) = \binom{n}{q}^{-1} \binom{n-k-l}{q-k} t^{-k} (1-t)^{-l} B_q^n(t), \quad (20)$$

into Eq. (18), we have

$$D_p^{(n,k,l)}(t; \alpha, \beta) = \sum_{q=k}^{n-l} \left\{ \binom{n-k-l}{p-k} \binom{n}{p}^{-1} \binom{n}{q}^{-1} \binom{n-k-l}{q-k} \times d_{p-k,q-k}^{(\alpha+2l,\beta+2k,n-k-l)} B_q^n(t) \right\}$$

then

$$d_{p,q}^{(k,l)} = \binom{n-k-l}{p-k} \binom{n}{p}^{-1} \binom{n}{q}^{-1} \binom{n-k-l}{q-k} d_{p-k,q-k}^{(\alpha+2l,\beta+2k,n-k-l)}.$$

Let

$$D^{(n,k,l)}(t) = \left(D_k^{(n,k,l)}(t; \alpha, \beta), \dots, D_{n-l}^{(n,k,l)}(t; \alpha, \beta) \right)^T,$$

$$B^{(n,k,l)}(t) = (B_k^n(t), \dots, B_{n-l}^n(t))^T$$

and $F^{(n,k,l)}(t) = (d_{p,q}^{(k,l)})_{p,q=k,\dots,n-l}$. Then Theorem 3 can be expressed as the following matrix form:

$$\begin{aligned} & \left(D^{(n,k,l)}(t), B^{(n,k,l)}(t) \right) = \\ & F^{(n,k,l)}(t) \left(B^{(n,k,l)}(t), B^{(n,k,l)}(t) \right) = I_{n-k-l+1}. \end{aligned} \quad (21)$$

From Theorem 1 and the properties of $k_{i,j}$, $\{B_i^n(t)\}_{i=k}^{n-l}$ can be expressed as linear combinations of $\{\beta_i(t; n, L)\}_{i=k}^{n-l}$:

$$B^{(n,k,l)}(t) = (A^{(n,k,l)}) \beta^{(n,k,l)}(t), \quad (22)$$

where

$$\begin{aligned} A^{(n,k,l)} &= (c_{i,j}^{(n)})_{i,j=k,\dots,n-l'} \\ \beta^{(n,k,l)}(t) &= (\beta_k(t; n, L), \dots, \beta_{n-l}(t; n, L))^T \end{aligned} \quad (23)$$

Substituting (22) into (21) and carrying out the proof

similar to that of Theorem 2, we obtain the dual basis functions of $\{\beta_i(t; n, L)\}_{i=k}^{n-l}$ in $P^{(n,k,l)}$.

Theorem 4. The dual basis $\{\gamma_i^{(k,l)}(t; n, L)\}_{i=k}^{n-l}$ of WSGB $\{\beta_i(t; n, L)\}_{i=k}^{n-l}$ in the space $P^{(n,k,l)}$ with respect to the Jacobi weight function has the following forms:

$$\gamma_i^{(k,l)}(t; n, L) = \sum_{j=k}^{n-l} k_{i,k}^{(n,k,l)} \beta_j(t; n, L), \quad i = k, \dots, n-l, \quad (24)$$

where

$$k_{i,k}^{(n,k,l)} = \sum_{p=k}^{n-l} \sum_{q=k}^{n-l} c_{p,i}^{(n)} d_{p,q}^{(k,l)} c_{q,j}^{(n)}, \quad i, j = k, \dots, n-l. \quad (25)$$

$c_{i,j}^{(n)}$, $d_{i,j}^{(k,l)}$ are defined by (4),(5),(6) and (17). Matrix $K^{(n,k,l)} = (k_{i,k}^{(n,k,l)})_{i,j=k,\dots,n-l}$ is defined as the dual matrix of WSGB $\{\beta_i(t; n, L)\}_{i=k}^{n-l}$.

IV. CONCLUSION

In this paper, by introducing the inner-product matrix of two vector functions and using conversion matrix between WSGB and Bernstein basis, we gave explicit formulas for the dual basis functions of Wang-Said type generalized Ball bases (WSGB) with and without boundary constraints with respect to the Jacobi weight function. This paper provides an efficient environment for the applications and extensions of the WSGB in CAGD.

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