A Robust PDE Based Image De-Noising Method

Nafis uddin Khan, K. V. Arya, and Manisha Pattanaik

***Abstract*—In this paper, a new method for noise reduction in images using partial differential equations (PDEs) has been proposed. In this approach, the method of noise reduction in spatial series using PDEs has been applied in spatial domain image. The image is initially represented as a spatial data matrix and the singular values of this matrix are used to divide the noisy image data into signal subspace and noise subspace. Since singular vectors are the span bases of the matrix, reducing the effect of noise from the singular vectors enhances the information embedded in the matrix. The proposed technique utilizes the PDEs for noise attenuation from the singular vectors. To evaluate the performance of the proposed method, a number of experiments has been performed on both real and synthetic images. The results indicate that the proposed method outperforms the existing approach in image de-noising.**

***Index Terms*—PDEs, diffusion equations, image de-noising, singular value decomposition, wiener filter.**

I. Introduction

In many image processing applications, a robust noise reduction is an important task. Hence, many researchers are investigating to develop a comprehensive noise reduction technique and many techniques have so far been developed to attain this goal. The nature and characteristics of noises changes significantly from application to application with position of the pixels of the image. It is therefore, very difficult to develop a versatile algorithm that works in diversified environments. Also, the objective of a noise reduction system may depend on the specific context and application. In image de-noising application, the original characteristics of the image are preserved along with improved signal to noise ratio (SNR). The ultimate goal of image de-noising is to improve an image in some predefined sense.

A number of methods for image de-noising have been presented in literatures [1]-[8]. In this paper, we focus on the use of partial differential equations (PDEs) for de-noising scientific images. These methods based on the partial differential equations have been extensively studied since the early work of Perona and Malik [1], [2] which gives a theoretical frame work and there is little information to guide a practitioner on the choice of various parameters used in the implementation of this method. As several authors have observed that work of Perona and Malik [1], [2] resulted in an ill-posed problem where images close to each other could produce divergent solutions and very different edges [7]. A common solution to this problem is to use a regularized (smoothed) version of the image to calculate the gradient as proposed by Malladi et al. [3]. One of the methods which have been widely used for signal de-noising in communication systems is the Wiener filter [5] commonly referred as the minimum mean square error filter. The Wiener filter was quite capable of essentially eliminating the blur in the image corrupted by white noise and has high SNR [12], [13]. In [9], [10], Hassanpour has introduced a method for signal de-noising using time-frequency distribution which is considerably effective for low SNR signals. However, this method can only be used for time-frequency distribution to reduce noise from time series. This technique is based on the singular value decomposition (SVD) of the matrix associated with the time-frequency representation of the signal [12] but the computational time is high. In image processing, if we apply this technique in spatial-frequency domain, a new image noise reduction and enhancement method can be evolved.

Recently, the spatial domain based approaches for noise reduction in images have received a considerable attention among researchers [6], [11]. These methods construct a spatial data matrix of the image and then classify each pixel of the image into two categories based on the mid scale image features contained in image gradient field. The classification results are then utilized to preset the parameters characterizing PDE based spatial method. In the proposed method, the spatial data matrix of noisy image is divided into signal subspace and noise subspace using the SVD based approach introduced by Hassanpour et al. [12]. Then using PDEs, noise from the singular vectors (SVs) are reduced and these SVs are used to reconstruct the matrix. This matrix with reduced noise is used to extract the spatial series representing the noise reduced image.

The rest of the paper is organized as follows: Section II discusses the Wiener filter. The proposed image de-noising method is presented in Section III. Section IV discusses the experimental results. In Section V, the performance evaluation is carried out and Section VI gives conclusion of the study.

II. Wiener Filter

Wiener filter [5], [13] has been widely used in signal processing for reduction of noise. In this method, the noisy signal is passed through a finite impulse response (FIR) filter whose coefficients are estimated by minimizing the mean square error between the clean signal and its estimate to restore the desired signal. This filter is one of the most fundamental approaches for noise reduction in image processing also which can be formulated on both spatial and frequency domains. The method is founded on considering images and noise as random processes and the objective is to

Manuscript received April 4, 2012; revised May 14, 2012.

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find an estimate $\hat{F}(x,y)$ of the uncorrupted image $F(x,y)$ such that the mean square error between them is minimized. This error measure is given by

$$e^2 = E\{\left(F(x,y) - \hat{F}(x,y)\right)^2\}$$

where E{.} is the mathematical expected value of the argument. It is assumed that the noise and the image are uncorrelated that one or other has Gaussian distribution with zero mean and that gray levels in the estimate are a linear function of the levels in the corrupted image. As mentioned before, this filter aims to minimize the total mean square error. Also this filter can only reduce the noise on signals if SNR value is higher than 4 and if SNR value is low then this filter can only reshape the noise.

III. Proposed Image De-noising Method

In this work, it is assumed that the original image is corrupted by an additive white Gaussian noise. For $F_n(i,j), i = 1 \dots\dots N, j = 1 \dots\dots M$ representing the noisy image whose data matrix H [12] is constructed as follows:

$$H = \begin{bmatrix} F_n(1,1) & \cdots & F_n(N,1) \\ \vdots & \ddots & \vdots \\ F_n(1,M) & \cdots & F_n(N,M) \end{bmatrix} \quad (1)$$

The singular value decomposition of matrix H with size P × Q is of the form:

$$H = UVW^T \quad (2)$$

where $U_{P\times r}$ and $W_{r\times Q}$ are orthogonal matrices and V is an $r \times r$ diagonal matrix of singular values with components $\sigma_{ij} = 0$ if $i \neq j$ and $\sigma_{ij} > 0$. Furthermore, it can be shown that $\sigma_{11} \geq \sigma_{22} \geq \cdots \geq 0$. The columns of the orthogonal matrices U and W are called the left and right Singular Vectors (SVs) respectively. The subspace separation introduced by Hassanpour et al. [8] has been used which is briefly expressed below:

$$H = UVW^T = (U_s \;\; U_n)\begin{bmatrix} V_s & 0 \\ 0 & V_n \end{bmatrix}\begin{pmatrix} W_s^T \\ W_n^T \end{pmatrix}$$

$$F_s = U_s U_s^T H = H W_s W_s^T$$

$$Q_n = U_n U_n^T H = H W_n W_n^T$$

where V_s and V_n represent the noise-free image subspace and noisy image subspace respectively. We must determine a threshold point in the V matrix in (2) where the lower singular values from the point can be categorized as noise subspace and hence, should be set to zero [8]. This threshold point can be determined by calculating the gradient of the image at each pixel position. It is observed that the noise subspace is mainly related to those singular values that are higher than the threshold point. Thus, these singular values are set to zero for space division.

By merely filtering the singular values, some noisy data like unrepresentative edges and curves will still be available in the image subspace. To further enhance the image information embedded in the image matrix, PDEs are applied to filter SVs of the image subspace matrix for reducing the noise effect.

Partial differential equations which have been initially introduced for the image processing application is the heat diffusion equation [1], [2]. The heat diffusion equation is defined as follows:

$$\frac{\partial F(x,y,t)}{\partial t} = \nabla\left(c(x,y,t)\nabla F(x,y,t)\right) \quad (3)$$

where $F(x,y,t)$ is the noisy image function, $c(x,y,t)$ is the diffusion coefficient and ∇ is the gradient operator.

In the method [1], [2], gradient in four directions of any pixels are calculated and then their diffusion coefficients are obtained to reduce the noise using (3). After number of iterations, the enhanced image is obtained.

In proposed method, the gradient of each pixel is computed using the pixels before and after the current pixel. Then, the diffusion coefficient in each directions of the current pixel, forward (c_f) and backward (c_b) are computed as follows:

$$F(x,y,t+\Delta t) = F(x,y,t) + \Delta t(\nabla_f c_f + \nabla_b c_b) \quad (4)$$

$$\nabla_f = [\left(F(x-\Delta x,y,t) - F(x,y,t)\right)^2 + (F(x,y-\Delta y,t) - F(x,y,t))^2]^{1/2} \quad (5)$$

$$c_f = \frac{1}{1+\left(\frac{\nabla_f}{k}\right)^2} \quad (6)$$

$$\nabla_b = [\left(F(x+\Delta x,y,t) - F(x,y,t)\right)^2 + (F(x,y+\Delta y,t) - F(x,y,t))^2]^{1/2} \quad (7)$$

$$c_b = \frac{1}{1+\left(\frac{\nabla_b}{k}\right)^2} \quad (8)$$

In (4)-(8), $F(x,y,t)$ is the noisy image, in this case it represents SVs, ∇_f and ∇_b are gradient magnitudes in forward and backward directions respectively, c_f and c_b are the corresponding diffusion coefficients for each of the directions, k is a constant value between 5 and 100, Δt is a coefficient between 0.1 to 0.3 representing the step of noise reduction in each iteration and Δx and Δy are the sampling rates in horizontal and vertical direction respectively.

The output vector is again applied to the algorithm at next iteration to gradually reduce the noise. This process is repeated for the number of iterations that lead to the best image. Then, the obtained vectors U and V are used to reconstruct the noise free image data matrix.

IV. Experimental Results and Analysis

To evaluate the performance of the proposed method, the experiments are performed on various synthetic and real images. These images are corrupted by additive Gaussian noise of standard deviation 20. The noisy images have been experimentally validated using Wiener filter and the proposed method. During the experimentation, different values of Δt and k are used. The two representative images of coins and Lena are shown in Fig.1. The restored images of coins and Lena after processed for noise reduction using Wiener filter and the proposed method are shown in Fig. 2 and 3 respectively.

Fig. 1. Two original test images of coins and lena respectively

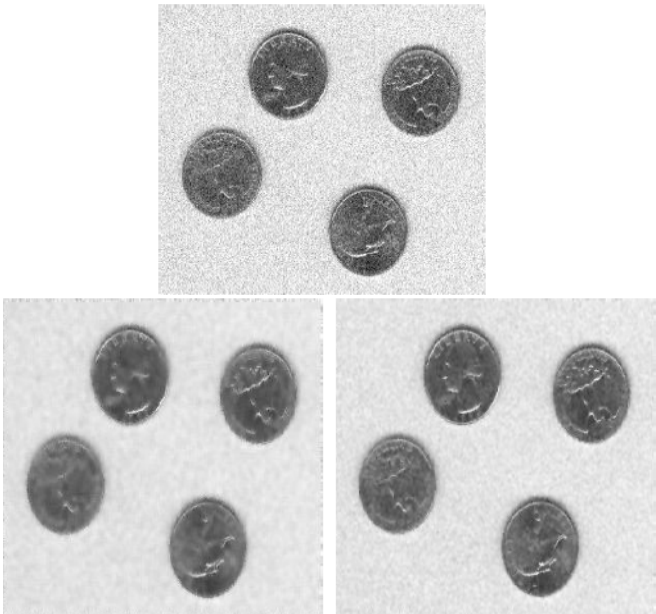

Fig. 2. (top) corrupted image of coins by gaussian noise of standard deviation 20; (bottom L to R) restored images by wiener filter and the proposed method

Fig. 3. (top) corrupted image of Lena by gaussian noise of standard deviation 20; (bottom L to R) Restored images by wiener filter and the proposed method

V. Performance Comparison and Evaluation

The proposed algorithm of noise reduction from image was implemented using Matlab7.1. We have used two measures to evaluate its performance. The results are then compared with those obtained using the Wiener filter. The performance measures used in this comparison are briefly described below:

A. Mean Square Error (MSE)

The mean square error (MSE) is frequently used in image processing and is defined as follows:

$$\boldsymbol{MSE} = \frac{1}{\mathbf{MN}}\sum_{\mathbf{i,j=1}}^{\mathbf{MN}}\{\boldsymbol{F_{original}}(\mathbf{i,j}) - \boldsymbol{F_{denoised}}(\mathbf{i,j})\}^2$$

Here $F_{original}$ is the original image and $F_{denoised}$ is the de-noised image. The smaller MSE value represents the better noise reduction algorithm.

B. Signal to Noise Ratio (SNR)

The signal to noise ratio (SNR) is a well known measure in image and signal processing. It is defined as follows:

$$(\boldsymbol{SNR})_{db} = \frac{\boldsymbol{Signal\ Power}}{\boldsymbol{Noise\ Power}}$$

$$= \mathbf{20}\boldsymbol{log}_{\mathbf{10}}\left(\frac{\boldsymbol{Max_F}}{\sqrt{\boldsymbol{MSE}}}\right)$$

Here, Max_F is the maximum possible pixel value of the image$F_{original}$. This criterion indicates the noise degradation in the enhanced image, the larger SNR value represents the enhanced image is closer to the original image.

The performance of the Wiener filter and the proposed approach has been compared by repeating experiments number of times and results are reported with average values of MSE and SNR for the two representative images in Table I and II respectively.

VI. Conclusion

In this study, a new approach is introduced for noise reduction in image. This approach is based on partial differential equations (PDEs). In this approach, image is at first represented as a spatial data matrix and then singular value decomposition (SVD) is used for space division. To reduce the effect of noise from singular vectors, PDEs are applied on the singular vectors of the matrix representing the image. The performance of the proposed method has been compared with that of Wiener filter based noise reduction method by computing MSE and SNR. It is observed through experimental results that the proposed method is robust and fast for image de-noising.

TABLE I: MSE and SNR Comparison for the Image of Coins

Performance Parameters	Noisy image	Wiener method	Proposed method
MSE	89.21	8.44	5.76
SNR	6.61	26.73	34.33

TABLE II: MSE and SNR Comparison for the Image of Lena

Performance Parameters	Noisy image	Wiener method	Proposed method
MSE	88.09	7.76	6.06
SNR	6.55	21.38	28.94

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